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Integrable geodesic flows and super polytropic gas equations

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Dedicated to Professor Alexander Alexandrovich Kirillov with great respect and admiration.

Abstract

The polytropic gas equations are shown to be the geodesic flows with respect to an L^2 metric on the semidirect product space $\mathrm{Diff}(S^1) \odot C^\infty(S^1)$, where $\mathrm{Diff}(S^1)$ is the group of orientation preserving diffeomorphisms of the circle. We also show that the $N = 1$ supersymmetric polytropic gas equation constitute an integrable geodesic flow on the extended Neveu–Schwarz space. Recently other kinds of supersymmetrizations have been studied vigorously in connection with superstring theory and are called supersymmetric-B (SUSY-B) extension. In this paper we also show that the SUSY-B extension of the polytropic gas equation form a geodesic flow on the extension of the Neveu–Schwarz space.

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1. Introduction

It is known that the periodic Korteweg–de Vries (KdV) equation can be interpreted as the geodesic flow of the right invariant metric on the Bott–Virasoro group, which at the identity is given by the L^2 inner product [27–29].

In the theory of integrable systems several multicomponent systems have been used sporadically. These are all biHamiltonian systems, enjoys a compatible pair of Hamiltonian structures. This system belongs to an infinite dimensional hierarchy of biHamiltonian systems. The resulting Hamiltonian flows can be mapped into each other by the recursion operator, which is formally defined as the “quotient” of the two Hamiltonian structure.

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Several of the well-known biHamiltonian systems that fall into the two component case are actually triHamiltonian. In our earlier papers [11–13], we show that several of these two component systems constitute geodesic flows on the extension of the Bott–Virasoro group.

In this paper we will consider first polytropic gas and Chaplygin gas (which corresponds to the specific choice of $\gamma = -1$ in the polytropic gas) equations. All these gas equations are integrable two component systems, it has an infinite number of conserved quantities, higher commuting flows and multiHamiltonian structure [25,26], and all of them constitute various integrable geodesic flows on the extended space of the group of diffeomorphisms of the circle with respect to an L^2 metric.

Recently, there has been a growing interest in supersymmetric integrable systems. These systems are much less understood than comparably ordinary integrable systems. Using the superconformal group with an L^2 metric, Ovsienko and Khesin [27] showed how the supersymmetric KdV equation arises from a geodesic flow. This result has been further extended by Devchand and Schiff [9]. They showed that the supersymmetric Camassa–Holm equation [6] describes geodesic motion on the superconformal group with respect to a metric induced by an H^1 metric [22]. Naturally, it is tempting to study the geodesic flow on the extension of the Neveu–Schwarz space [19–21]. This would yield a supersymmetric generalization of the two component integrable systems.

Recently in an interesting paper Das and Popowicz [8] initiated the study of supersymmetric generalization of the polytropic gas equation. This equation is related to various theories, viz. the Born–Infeld system [2], D-Brane theory [4,15], Monge–Ampére equation [23,24], etc. Jackiw and coworkers ([4,15] and references therein) demonstrated that a supersymmetrization of the Chaplygin gas can be obtained from the superstring theory in three dimensions with a κ supersymmetry after gauge fixing. The supersymmetric system of equations obtained in this form are called supersymmetric-B (SUSY-B) extension. In this paper we will show that this set of equations also arises from a geodesic flow on the superconformal group with respect to an L^2 metric.

Following Ebin–Marsden [10], we enlarge $\mathrm{Diff}(S^1)$ to a Hilbert manifold $\mathrm{Diff}^s(S^1)$, the diffeomorphisms of the Sobolev class H^s. This is a topological space. If $s > n/2$, it makes sense to talk about an H^s map from one manifold to another. Using local chart one can check that the derivations of order less than or equal to s are square integrable.

In Section 2, at first we will review some of the essential features of $\mathrm{Diff}(S^1) \odot C^\infty(S^1)$ and its algebra and then we will study the polytropic gas equation as a geodesic flow on the $\mathrm{Diff}^s(S^1) \odot C^\infty(S^1)$, related to L^2 inner product. In Section 3, after a brief discussions on the extension of the Neveu–Schwarz space we will go on to show that the supersymmetric polytropic gas system is the geodesic flow on this space. We also present the SUSY-B extension of the polytropic gas equation as the geodesic flow. Thus, we present a mathematical rigorous description of SUSY-B polytropic gas equation. Section 4 is devoted to multicomponent generalization of the polytropic gas equation. We end with a brief conclusion in Section 5.

2. Background: polytropic gas equation and geodesic flow

Let $\mathrm{Diff}^s(S^1)$ be the group of orientation preserving Sobolev H^s diffeomorphisms of the circle. It is known that the group $\mathrm{Diff}^s(S^1)$ as well as its Lie algebra of vector fields on S^1, $T_{\rm id}\,\mathrm{Diff}^s(S^1) = \mathrm{Vect}^s(S^1)$, have non-trivial one-dimensional central extensions, the Bott–Virasoro group $\mathrm{Diff}^s(S^1)$ and the Virasoro algebra Vir, respectively [16,17,28].

The Lie algebra $\mathrm{Vect}^s(S^1)$ is the algebra of smooth vector fields on S^1. This satisfies the commutation relations

$$\left[f\frac{\mathrm{d}}{\mathrm{d}x}, g\frac{d}{\mathrm{d}x}\right] := (f(x)g'(x) - f'(x)g(x))\frac{\mathrm{d}}{\mathrm{d}x}. \tag{1}$$

One parameter family of $\mathrm{Vect}^s(S^1)$ acts on the space of smooth functions $C^\infty(S^1)$ by

$$L^{(\mu)}_{f(x)(\mathrm{d}/\mathrm{d}x)}(x) = f(x)'(x) - \mu f'(x)(x), \tag{2}$$

where

$$L^{(\mu)}_{f(x)(\mathrm{d}/\mathrm{d}x)}(x) = f(x)\frac{\mathrm{d}}{\mathrm{d}x} - \mu f'(x)$$

is the derivative with respect to the vector field $f(x)(\mathrm{d}/\mathrm{d}x)$ and satisfies

$$[L^{(\mu)}_{f(x)(\mathrm{d}/\mathrm{d}x)}, L^{(\mu)}_{g(x)(\mathrm{d}/\mathrm{d}x)}] = L^{(\mu)}_{(f(x)g'(x)-f'(x)g(x))(\mathrm{d}/\mathrm{d}x)}.$$

The Lie algebra of $\mathrm{Diff}^s(S^1) \odot C^\infty(S^1)$ is the semidirect product Lie algebra

$$\hat{\mathcal{G}} = \mathrm{Vect}^s(S^1) \odot C^\infty(S^1).$$

An element of $\hat{\mathcal{G}}$ is a pair $(f(x)(\mathrm{d}/\mathrm{d}x), a(x))$, where $f(x)(\mathrm{d}/\mathrm{d}x) \in \mathrm{Vect}^s(S^1)$ and $a(x) \in C^\infty(S^1)$.

The extension of the Lie algebra $\hat{\mathcal{G}}$ is given by

$$\mathcal{G} = \mathrm{Vect}^s(S^1) \odot C^\infty(S^1) \oplus \mathbf{R}^3.$$

This has been considered in various places [1,14,19]. It was shown in [22] that the cocycles define the universal central extension of the Lie algebra $\mathrm{Vect}^s(S^1) \odot C^\infty(S^1)$. This means $H^2(\mathrm{Vect}(S^1) \odot C^\infty(S^1)) = \mathbf{R}^3$.

Definition 1. The commutation relation in $\hat{\mathcal{G}}$ is given by

$$\left[\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b(x)\right)\right] := \left((fg' - f'g)\frac{\mathrm{d}}{\mathrm{d}x}, fb' - ga'\right). \tag{3}$$

The dual space of smooth functions $C^\infty(S^1)$ is the space of distributions (generalized functions) on S^1, of particular interest are the orbits in $\hat{\mathcal{G}}^*_{\mathrm{reg}}$. In the case of current group, Gelfand, Vershik and Graev have constructed some of the corresponding representations.

Definition 2. The regular part of the dual space $\hat{\mathcal{G}}^*$ to the Lie algebra $\hat{\mathcal{G}}$ is defined as follows: Consider

$$\hat{\mathcal{G}}^*_{\mathrm{reg}} = C^\infty(S^1) \oplus C^\infty(S^1),$$

and fix the pairing between this space and $\hat{\mathcal{G}}$, $\langle\cdot,\cdot\rangle : \hat{\mathcal{G}}^*_{\rm reg} \otimes \hat{\mathcal{G}} \to \mathbf{R}$:

$$\langle \hat{u}, \hat{f}\rangle = \int_{S^1} f(x)u(x)\,\mathrm{d}x + \int_{S^1} (x)v(x)\,\mathrm{d}x, \tag{4}$$

where $\hat{u} = (u(x), v)$ and $\hat{f} = (f(\mathrm{d}/\mathrm{d}x), a)$.

Let us extend (4) to a right invariant metric on the semidirect product space $\mathrm{Diff}^s(S^1) \odot C^\infty(S^1)$ by setting

$$\langle \hat{u}, \hat{f}\rangle_{\hat{\xi}} = \langle d_{\hat{\xi}} R_{\hat{\xi}-1}\hat{u}, d_{\hat{\xi}} R_{\hat{\xi}-1}\hat{f}\rangle_{L^2} \tag{5}$$

for any $\hat{\xi} \in \mathcal{G}$ and $\hat{u}, \hat{f} \in T_{\hat{\xi}}\mathcal{G}$, where

$$R_{\hat{\xi}} : \mathcal{G} \to \mathcal{G}$$

is the right translation by $\hat{\xi}$.

At first we shall show that the polytropic gas equation is precisely the Euler–Arnold equation [3] on the dual space of $\mathcal{G}$ associated with the L^2 inner product.

Given any three elements

$$\hat{f} = \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right), \qquad \hat{g} = \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\right), \qquad \hat{u} = \left(u\frac{\mathrm{d}}{\mathrm{d}x}, v\right)$$

in $\mathcal{G}$, we obtain the following lemma.

Lemma 1.

$$\mathrm{ad}^*_{\hat{f}}\hat{u} = \begin{pmatrix} 2f'(x)u(x) + f(x)u'(x) + a'(x)v(x) \\ f'(x)v(x) + f(x)v'(x) \end{pmatrix}.$$

Proof. This follows from

$$\begin{aligned}\langle \mathrm{ad}^*_{\hat{f}}\hat{u}, \hat{g}\rangle_{L^2} &= \langle \hat{u}, [\hat{f},\hat{g}]\rangle_{L^2} = \left\langle \left(u(x)\frac{\mathrm{d}}{\mathrm{d}x}, v(x), c\right), \left[(fg' - f'g)\frac{\mathrm{d}}{\mathrm{d}x}, fb' - ga', \omega\right]\right\rangle_{L^2} \\ &= -\int_{S^1} (fg' - f'g)u(x)\,\mathrm{d}x - \int_{S^1} (fb' - ga')v\,\mathrm{d}x.\end{aligned}$$

Since f, g, u are periodic functions, hence integrating by parts we obtain

$$\mathrm{RHS} = \begin{pmatrix} 2f'(x)u(x) + f(x)u'(x) + a'(x)v(x) \\ f'(x)v(x) + f(x)v'(x) \end{pmatrix}.$$

□

The implectic or Poisson operator is given by

$$\mathcal{O} = \begin{pmatrix} \partial u + u\partial & v\partial & 0 & 0 \\ \partial v & 0 & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & v & 0 \end{pmatrix}. \tag{6}$$

The Euler–Arnold equation is the Hamiltonian flow on the coadjoint orbit in $\hat{\mathcal{G}}^*$ [3], generated by the Hamiltonian

$$H(u, v) = \frac{1}{2}\left(u^2 + \frac{v^\gamma}{\gamma(\gamma-1)}\right), \tag{7}$$

given by

$$\begin{pmatrix} \dfrac{\delta H}{\delta u} \\ \dfrac{\delta H}{\delta v} \end{pmatrix}_t = \mathcal{O}\begin{pmatrix} \dfrac{\delta H}{\delta u} \\ \dfrac{\delta H}{\delta v} \end{pmatrix}. \tag{8}$$

Let V be a vector space and assume that the Lie group G acts on the left by linear maps on V, thus G acts on the left on its dual space V^* (for details, see for example, [7]).

Proposition 1. *Let $G \odot V$ be a semidirect product space (possibly infinite dimensional), equipped with a metric $\langle\cdot,\cdot\rangle$ which is right translation. A curve $t \to c(t)$ in $G \odot V$ is a geodesic of this metric if and only if $u(t) = d_{c(t)} R_{c(t)^{-1}} \dot{c}(t)$ satisfies the Euler–Arnold equation.*

Thus, we obtain the polytropic gas equation.

Theorem 1. *Let $t \mapsto c$ be a curve in the $\mathrm{Diff}^s(S^1) \odot C^\infty(S^1)$. Let $c = (e, e)$ be the initial point, directing to the vector $c(0) = (u(x)(\mathrm{d}/\mathrm{d}x), v(x))$. Then $c(t)$ is a geodesic of the L^2 metric if and only if $(u(x,t)(\mathrm{d}/\mathrm{d}x), v(x,t))$ satisfies the polytropic gas equation*

$$u_t = 3uu_x + v^{\gamma-2}v_x, \qquad v_t = (uv)_x.$$

Remark 1 ([5]). The Chaplygin gas equation

$$u_t + uu_x + v^{-3}v_x = 0, \qquad v_t + (uv)_x = 0, \tag{9}$$

transforms under $u(x) = -(b_x/a_x)$ and $v(x) = a_x$ to (A) some minimal surface equation

$$a_{xx}(a_t^2 - 1) - 2a_x a_t a_{xt} + a_x^2 a_{tt} = 0, \tag{10}$$

and (B) Monge–Ampère equation

$$U_{xt}^2 - U_{xx}U_{tt} = 1. \tag{11}$$

Proof. By direct substitution the first and second equations become

$$a_{xx}(1 - b_x^2) + 2a_x b_x b_{xx} - a_x^2 b_{xt} = 0,$$

and $a_t = b_x$, respectively. We obtain the first part.

The equation $a_t = b_x$ allows us to write a and b in terms of a potential U, such that

$$a = U_x, \qquad b = U_t.$$

Thus, we obtain Monge–Ampére equation. □

3. Geodesic flow and super polytropic gas equations

The first and foremost characteristic property of super algebra is that all the additive groups of its basic and derived structures are $\mathbb{Z}_2$ graded. A vector superspace is a $\mathbb{Z}_2$ graded vector space $V = V_0 + V_1$. An element v of V_0 (resp. V_1) is said to be even (resp. odd). The super commutator of a pair of elements $v, \omega \in V$ is defined to be the element

$$[v, w] = vw - (-1)^{\bar{v}\bar{w}} wv,$$

whee $\bar{v}$ and $\bar{w}$ are the degrees of v and w respectively.

The generalized Neveu–Schwarz superalgebra has two parts, Bosonic (even) and Fermionic (odd). These are given by

$$S\hat{\mathcal{G}}_{\rm B} = \mathrm{Vect}(S^1) \oplus C^\infty(S^1), \tag{12}$$

$$S\hat{\mathcal{G}}_{\rm F} = C^\infty(S^1) \oplus C^\infty(S^1). \tag{13}$$

There are three different actions:

(A) Action of Bosonic part on Bosonic part, given in (7).
(B) Action of Bosonic part on Fermionic part, given by

$$\left[\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right), (\phi(x), \alpha(x))\right] := \begin{pmatrix} f(x)\phi' - \frac{1}{2}f'(x)\phi(x) \\ f(x)\alpha'(x) + \frac{1}{2}f'(x)\alpha(x) - \frac{1}{2}a'(x)\phi(x) \end{pmatrix}. \tag{14}$$

(C) Action of Fermionic part on Fermionic part, given by

$$[\cdot,\cdot]_+ : S\hat{\mathcal{G}}_{\rm F} \otimes S\hat{\mathcal{G}}_{\rm F} \to S\hat{\mathcal{G}}_{\rm B},$$

$$[(\phi(x), \alpha(x)), (\psi(x), \beta(x))]_+ = \left(\phi\psi\frac{\mathrm{d}}{\mathrm{d}x}, \phi\beta + \alpha\psi\right). \tag{15}$$

Definition 3. The pairing between the regular part of the dual space $S\hat{\mathcal{G}}^*$ and $S\hat{\mathcal{G}}$ is given by

$$\left\langle (u(x), v(x), \psi(x), \beta), \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x), \phi(x), \alpha\right)\right\rangle = \int_{S^1} f(x)u(x)\,\mathrm{d}x + \int_{S^1} a(x)v(x)\,\mathrm{d}x + \int_{S^1}\phi(x)\psi(x)\,\mathrm{d}x + \int_{S^1}\alpha(x)\beta(x)\,\mathrm{d}x. \tag{16}$$

Lemma 2.

$$\mathrm{ad}^*_{\hat{f}}\hat{u} = \begin{pmatrix} 2uf'(x) + u'f + u'v + \frac{1}{2}\eta'\phi\frac{3}{2}\eta\phi' + \frac{1}{2}(\xi u - \xi' u) \\ f'v + fv' + \frac{1}{2}(\beta'\phi + \beta\phi') \\ f\eta' + \frac{3}{2}f'\eta + \frac{1}{2}\alpha'\xi + \kappa(u\phi + v\alpha) + 2\phi'' \\ f\xi' + \frac{1}{2}f'\xi + \kappa v\phi \end{pmatrix}. \tag{17}$$

Sketch of the proof. Using the definition of the coadjoint action

$$\langle \mathrm{ad}^*_{\hat{f}}\hat{u}, \hat{g}\rangle = \langle \hat{f}, [\hat{u}, \hat{g}]\rangle,$$

where

$$\hat{f} = \begin{pmatrix} f(x) \\ a(x) \\ \phi(x) \\ \alpha(x) \end{pmatrix}, \qquad \hat{u} = \begin{pmatrix} u(x) \\ v(x) \\ \psi(x) \\ \beta(x) \end{pmatrix}, \qquad \hat{g} = \begin{pmatrix} g(x) \\ b(x) \\ \chi(x) \\ \gamma(x) \end{pmatrix},$$

we obtain

$$\left\langle (u, v, \eta, \xi), \begin{pmatrix} (fg' - f'g)\dfrac{\mathrm{d}}{\mathrm{d}x} + \kappa\phi\chi\dfrac{\mathrm{d}}{\mathrm{d}x} \\ fb' - f'b + \kappa(\phi\gamma + \alpha\chi) \\ f\chi' - \frac{1}{2}f'\chi + g\phi' - \frac{1}{2}g'\phi \\ f\gamma' + \frac{1}{2}f'\gamma - \frac{1}{2}a'\gamma + g\alpha' + \frac{1}{2}g'\alpha - \frac{1}{2}b'\phi \end{pmatrix} \right\rangle.$$

Our result follows from this formula.

3.1. *Case 1: N = 1 supersymmetric polytropic gas*

In this case, we will consider $\kappa = 0$, that is, we will not consider the part arising from the action of Fermionic part on Fermionic part. The implectic operator for this system is

$$\mathcal{O}_{\mathrm{SUSY}} = \begin{pmatrix} \partial u + u\partial & v\partial & \frac{3}{2}\eta\partial + \frac{1}{2}\eta' & \frac{1}{2}\xi\partial - \frac{1}{2}\xi' \\ \partial v & 0 & -\frac{1}{2}\partial\xi & 0 \\ \frac{3}{2}\eta\partial + \eta' & \frac{1}{2}\xi' & 0 & 0 \\ \xi' + \frac{1}{2}\xi\partial & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

Now we use the Euler–Arnold equation:

$$\begin{pmatrix} u \\ v \\ \eta \\ \xi \end{pmatrix}_t = \mathcal{O}_{\mathrm{SUSY}} \begin{pmatrix} \dfrac{\delta H}{\delta u} \\ \dfrac{\delta H}{\delta v} \\ \dfrac{\delta H}{\delta \eta} \\ \dfrac{\delta H}{\delta \xi} \end{pmatrix}. \tag{19}$$

If we take Hamiltonian H such that

$$\frac{\delta H}{\delta u} = u, \qquad \frac{\delta H}{\delta v} = \frac{2}{\gamma}u^{\gamma-2}, \qquad \frac{\delta H}{\delta \eta} = \eta', \qquad \frac{\delta H}{\delta \xi} = -\frac{2(\gamma-2)}{\gamma}\xi'.$$

Thus, we obtain the supersymmetric version of the polytropic gas equation as a geodesic flow on the superconformal group.

Theorem 2. *Let* $t \mapsto c'$ *be a curve in the* $\mathrm{SDiff}^s(S^1) \odot C^\infty(S^1)$. *Let* $c = (e, e)$ *be the initial point, directing to the vector* $c'(0) = (u(x)(\mathrm{d}/\mathrm{d}x), v(x), \eta(x), \xi(x))$. *Then* $c'(t)$ *is a geodesic of the* L^2 *metric if and only if* $(u(x,t)(\mathrm{d}/\mathrm{d}x), v(x,t), \eta(x,t), \xi(x,t))$ *satisfies the polytropic gas equation*

$$u_t = 3uu_x + \frac{2}{\gamma} v_x v^{\gamma-2} - \frac{\gamma-2}{\gamma}\xi\xi_{xx}v^{\gamma-3} - \frac{(\gamma-2)(\gamma-3)}{\gamma}v_x v^{\gamma-4} + \eta\eta_{xx}, \tag{20}$$

$$v_t = (uv - \tfrac{1}{2}\xi\eta_x)_x, \tag{21}$$

$$\eta_t = \eta_x u + \frac{3}{2}\eta u_x + \frac{\gamma-2}{\gamma}\xi v_x v^{\gamma-3}, \tag{22}$$

$$\xi_t = \xi_x u + \tfrac{1}{2}\xi u_x. \tag{23}$$

This equation is closely related to what Das and Popowicz [8] obtained for the supersymmetric polytropic equation.

Let us study some *special cases* of these equations.

(A) If we set the super variables $\xi = \eta = 0$, we get back the polytropic gas equation.
(B) If we set $v = \xi = 0$, we obtain

$$u_t = 3uu_x + \eta\eta_{xx}, \qquad \eta_t = \eta_x u + \tfrac{3}{2}\eta u_x.$$

This is a Fermionic extension of the dispersionless KdV equation. Modulo rescalings, this is related to the super dispersionless KdV of Mathieu and Manin–Radul type (see for example, [9]).

3.2. *Case 2: SUSY-B extension of the polytropic gas equations*

The supersymmetric Bosonic extensions are simple supersymmetrizations of a Bosonic integrable model that are automatically integrable. The basic idea is to supersymmetrize the ordinary fields. These supersymmetric system of equations has the peculiar feature that the two Bosonic equations do not have any Fermion terms. Mathematically speaking, we will only consider the part arises from the action of Fermion part on Fermion part. Earlier such type of supersymmetrizations were not considered seriously. These are considered trivial extensions. Due to the advancements of superstring theory, especially D-Brane theory, these type of supersymmetrizations are becoming important.

In this section we will show that this set of equations also can be manifested as a geodesic flow on superconformal group.

Lemma 3.

$$\mathrm{ad}^*_{\hat{f}}\hat{u} = \begin{pmatrix} 2uf'(x) + u'f + a'v \\ f'v + fv' \\ u\phi + v\alpha \\ v\phi \end{pmatrix}.$$

Thus, the implectic operator for the SUSY-B extension of the polytropic gas equation is

$$\mathcal{O}_{\text{SUSY-B}} = \begin{pmatrix} \partial u + u\partial & u\partial & 0 & 0 \\ \partial v & 0 & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & v & 0 \end{pmatrix}. \tag{24}$$

If we take Hamiltonian H such that

$$\frac{\delta H}{\delta u} = u, \qquad \frac{\delta H}{\delta v} = v^{\gamma-2}, \qquad \frac{\delta H}{\delta \eta} = \eta', \qquad \frac{\delta H}{\delta \xi} = \xi' v^{\gamma-3}.$$

Hence, we obtain the following theorem.

Theorem 3. *The SUSY-B extension of the polytropic gas equations*

$$u_t = 3uu_x + v_x v^{\gamma-2}, \tag{25}$$

$$v_t = (uv)_x, \tag{26}$$

$$\eta_t = \eta_x u + \xi_x v^{\gamma-3}, \tag{27}$$

$$\xi_t = v\eta_x, \tag{28}$$

constitute the Euler–Arnold flows on the dual space of $\text{SVect}(S^1) \odot C^\infty(S^1)$.

3.3. Kac–Moody algebra and Lie–Poisson structure

The Hamiltonian operators associated with polytropic gas equations give rise to Kac–Moody algebras. There is an explicit algorithm for the construction of Kac–Moody algebras from the Hamiltonian operator which is essentially based on Fourier analysis.

The Hamiltonian operator of the polytropic gas is

$$\mathcal{O} = \begin{pmatrix} \partial u + u\partial & v\partial \\ \partial v & 0 \end{pmatrix}.$$

Let us calculate the Lie–Poisson brackets of $u(x)$ and $v(x)$:

$$\{u(x), u(x')\} = 2u\delta'(x - x') + u'\delta(x - x'), \tag{29}$$

$$\{u(x), v(x')\} = v\delta'(x - x'), \tag{30}$$

$$\{v(x), v(x')\} = \delta'(x - x'). \tag{31}$$

Let us Fourier expand $u(x)$ and $v(x)$

$$u(x) = \sum_{p=1}^{\infty} L_p\, \mathrm{e}^{\mathrm{i}px} + \alpha, \qquad v(x) \sum_{p=1}^{\infty} S_p\, \mathrm{e}^{\mathrm{i}px} + \beta.$$

Hence, we obtain the Kac–Moody algebra corresponding to the above Poisson brackets

$$[L_n, L_m] = (n - m)L_{n+m} + \alpha\delta_{n+m,0}, \tag{32}$$

$$[L_n, S_m] = -mS_{n+m} + \beta\delta_{n+m,0}, \tag{33}$$

$$[S_n, S_m] = n\delta_{n+m,0}. \tag{34}$$

Similarly, one can obtain super Kac–Moody algebra from (16).

4. Multicomponent polytropic gas equation

In this section we consider the Euler–Arnold equation on the dual space of $\mathrm{Vect}(S^1) \odot C^\infty(S^1)^{k+1}$. We split the space $C^\infty(S^1)^{k+1}$ into $C^\infty(S^1) \times C^\infty(S^1)^k$. A typical element of $\mathrm{Vect}(S^1) \odot C^\infty(S^1)^{k+1}$ is

$$\begin{pmatrix} f(x)\dfrac{\mathrm{d}}{\mathrm{d}x} \\ a(x) \\ \vec{p} \end{pmatrix}.$$

This type of construction has been discussed by Kupershmidt [18].

Definition 4. The commutation relation in $\hat{\mathcal{G}}$ is given by

$$\left[\begin{pmatrix} f(x)\dfrac{\mathrm{d}}{\mathrm{d}x} \\ a(x) \\ \vec{p} \end{pmatrix}, \begin{pmatrix} g(x)\dfrac{\mathrm{d}}{\mathrm{d}x} \\ b(x) \\ \vec{q} \end{pmatrix} \right] := \begin{pmatrix} (fg' - f'g)\dfrac{\mathrm{d}}{\mathrm{d}x} \\ fb' - ga' \\ f\vec{q} - g\vec{p} \end{pmatrix}. \tag{35}$$

Definition 5. The regular part of the dual space $\hat{\mathcal{G}}^*$ to the Lie algebra $\hat{\mathcal{G}}$ as follows. Consider

$$\hat{\mathcal{G}}^*_{\mathrm{reg}} = C^\infty(S^1) \oplus \underbrace{C^\infty(S^1) \oplus \cdots \oplus C^\infty(S^1)}_{k},$$

and fix the pairing between this space and $\hat{\mathcal{G}}$, $\langle\cdot,\cdot\rangle : \hat{\mathcal{G}}^*_{\mathrm{reg}} \otimes \hat{\mathcal{G}} \to \mathbf{R}$:

$$\langle \hat{u}, \hat{f} \rangle = \int_{S^1} f(x)u(x)\,\mathrm{d}x + \int_{S^1} u(x)\vec{v}(x)\,\mathrm{d}x + \vec{w}\gamma, \tag{36}$$

where $\hat{u} = (u(x), v, \vec{w}, \gamma)$ and $\hat{f} = (f(\mathrm{d}/\mathrm{d}x), a, \vec{p}, \alpha)$.

Again from the coadjoint action, we obtain the following set of integrable Hamiltonian system:

$$u_t = 3uu_x + v^{\gamma-2} = 0, \qquad v_t = (vu)_x, \qquad \vec{w}_t = (\vec{w}u)_x.$$

This is another *avatar* of polytropic gas equation.

5. Outlook

In this paper we have examined various $N = 1$ supersymmetric extension of the polytropic gas equations. In particular, we have studied that these equations are associated to the coadjoint orbits of the extended Neveu–Schwarz group.

Our work provides a further instance of integrability arising in the setting of geodesic flow on a super group manifold. Further investigation is needed in order to determine whether the various super polytropic gas equations are integrable irrespective of the choice of various Grassman algebra.

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